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Research Article

COMMON FIXED POINT THEOREM FOR OCCASIONALLY WEAKLY COMPATIBLE MAPPINGS IN FUZZY METRIC SPACE

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ABSTRACT

The present paper deals with common fixed point theorem in fuzzy metric space by employing the notion of occasionally weakly compatible mappings. Our result generalizes the recent result of Singh et. al. [16].

Keywords: Common fixed points, fuzzy metric space, compatible maps, occasionally weakly compatible mappings and weak compatible mappings.

INTRODUCTION

In 1965, Zadeh [17] introduced the concept of Fuzzy set as a new way to represent vagueness in our everyday life. However, when the uncertainty is due to fuzziness rather than randomness, as sometimes in the measurement of an ordinary length, it seems that the concept of a fuzzy metric space is more suitable. We can divide them into following two groups: The first group involves those results in which a fuzzy metric on a set X is treated as a map where X represents the totality of all fuzzy points of a set and satisfy some axioms which are analogous to the ordinary metric axioms. Thus, in such an approach numerical distances are set up between fuzzy objects. On the other hand in second group, we keep those results in which the distance between objects is fuzzy and the objects themselves may or may not be fuzzy. In this paper we deal with the Fuzzy metric space defined by Kramosil and Michalek [10] and modified by George and Veeramani [4]. Recently, Grabiec [5] has proved fixed point results for Fuzzy metric space. In the sequel, Singh and Chauhan [13] introduced the concept of compatible mappings in Fuzzy metric space and proved the common fixed point theorem. Jungck et. al. [8] introduced the concept of compatible maps of type (A) in metric space and proved fixed point theorems. Cho [2, 3] introduced the concept of compatible maps of type (α) and compatible maps of type (β) in fuzzy metric space. In 2011, using the concept of compatible maps of type (A) and type (β), Singh et. al. [14, 15] proved fixed point theorems in a fuzzy metric space. Recently in 2012, Jain et. al. [6, 7] and Sharma et. al. [12] proved various fixed point theorems using the concepts of semi-compatible mappings, property (E.A.) and absorbing mappings. In this paper, a fixed point theorem for six self maps has been established using the concept of occasionally weak compatible maps which generalizes the result of Singh et. al. [16]. For the sake of completeness, we recall some definitions and known results in Fuzzy metric space.

2,Preliminaries

Definition 2.1. [11] A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is called a *t-norm* if ([0, 1], *) is an abelian topological monoid with unit 1 such that $a * b \le c * d$ whenever $a \le c$ and $b \le d$ for $a, b, c, d \in [0, 1]$.

Examples of t-norms are $a * b = ab$ and $a * b = \min\{a, b\}$.

Definition 2.2. [11] The 3-tuple (X, M, *) is said to be a *Fuzzy metric space* if X is an arbitrary set, * is a continuous t-norm and M is a Fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions :

for all $x, y, z \in X$ and $s, t > 0$.

(FM-1) $M(x, y, 0) = 0$,

(FM-2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$,

(FM-3) $M(x, y, t) = M(y, x, t)$,

(FM-4) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$,

(FM-5) $M(x, y, .) : [0, \infty) \to [0, 1]$ is left continuous,

(FM-6) $\lim_{t\to\infty} M(x, y, t) = 1$.

Note that M(x, y, t) can be considered as the degree of nearness between x and y with respect to t. We identify $x = y$ with $M(x, y, t) = 1$ for all $t > 0$. The following example shows that every metric space induces a Fuzzy metric space.

Example 2.1. [11] Let (X, d) be a metric space. Define $a * b = \min\{a, b\}$ and $M(x, y, t) = \dfrac{t}{t + d(x, y)}$ for all $x, y \in X$ and all $t > 0$.

Then (X, M, *) is a Fuzzy metric space. It is called the Fuzzy metric space induced by d.

Definition 2.3. [11] A sequence $\{x_n\}$ in a Fuzzy metric space (X, M, *) is said to be a *Cauchy sequence* if and only if for each $\varepsilon > 0$, $t > 0$, there exists $n_0 \in N$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \ge n_0$.

The sequence $\{x_n\}$ is said to *converge* to a point x in X if and only if for each $\varepsilon > 0$, $t > 0$ there exists $n_0 \in N$ such that $M(x_n, x, t) > 1 - \varepsilon$ for all $n \ge n_0$.

A Fuzzy metric space (X, M, *) is said to be *complete* if every Cauchy sequence in it converges to a point in it.

Definition 2.4. [13] Self mappings A and S of a Fuzzy metric space (X, M, *) are said to be *compatible* if and only if $M(ASx_n, SAx_n, t) \to 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $Sx_n, Ax_n \to p$ for some p in X as $n \to \infty$.

Definition 2.5. [14] Two self maps A and B of a fuzzy metric space (X, M, *) are said to be weak compatible if they commute at their coincidence points, i.e. $Ax = Bx$ implies $ABx = BAx$.

Definition 2.6. Self maps A and S of a Fuzzy metric space (X, M, *) are said to be occasionally weakly compatible (owc) if and only if there is a point x in X which is coincidence point of A and S at which A and S commute.

Proposition 2.1. [15] In a fuzzy metric space (X, M, *) limit of a sequence is unique.

Proposition 2.2. [13] Let S and T be compatible self maps of a Fuzzy metric space (X, M, *) and let $\{x_n\}$ be a sequence in X such that Sx_n, $Tx_n \to u$ for some u in X. Then $STx_n \to Tu$ provided T is continuous.

Proposition 2.3. [13] Let S and T be compatible self maps of a Fuzzy metric space (X, M, *) and Su = Tu for some u in X then

$$\mathbf{STu = TSu = SSu = TTu.}$$

Lemma 2.1. [5] Let (X, M, *) be a fuzzy metric space. Then for all x, y $\in$ X, M(x, y, .) is a non-decreasing function.

Lemma 2.2. [1] Let (X, M, *) be a fuzzy metric space. If there exists k $\in$ (0, 1) such that for all x, y $\in$ X

$$M(x, y, kt) \geq M(x, y, t)\ \forall\ t > 0$$

then x = y.

Lemma 2.3. [15] Let $\{x_n\}$ be a sequence in a fuzzy metric space (X, M, *). If there exists a number k $\in$ (0, 1) such that

$M(x_{n+2}, x_{n+1}, kt) \geq M(x_{n+1}, x_n, t)\ \forall\ t > 0$ and $n \in N$.

Then $\{x_n\}$ is a Cauchy sequence in X.

Lemma 2.4.[9] The only t-norm * satisfying r * r ≥ r for all r $\in$ [0, 1] is the minimum t-norm, that is

a * b = min {a, b} for all a, b $\in$ [0, 1].

3.Main Result.

Theorem 3.1. Let (X, M, *) be a complete Fuzzy metric space with continuous t-norm * and t * t ≥ t, for all t $\in$ [0, 1] and let A, B, S, T, P and Q be mappings from X into itself such that the following conditions are satisfied:

(a) $P(X) \subset ST(X)$, $Q(X) \subset AB(X)$;
(b) AB = BA, ST = TS, PB = BP, QT = TQ;
(c) either P or AB is continuous;
(d) (P, AB) is compatible and (Q, ST) is occasionally weakly compatible;
(e) There exists k $\in$ (0, 1) such that $\forall$ x, y $\in$ X and t > 0,
M(Px, Qy, kt) ≥ Min {M(Qy, STy, t), M(ABx, STy, t), M(Px, ABx, t)}.

Then A, B, S, T, P and Q have a unique common fixed point in X.

Proof. Let $x_0 \in X$. From (a) there exist $x_1, x_2 \in X$ such that

$Px_0 = STx_1$ and $Qx_1 = ABx_2$.

Inductively, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$Px_{2n-2} = STx_{2n-1} = y_{2n-1}$ and $Qx_{2n-1} = ABx_{2n} = y_{2n}$ for n = 1, 2, 3, ...

Step 1. Put $x = x_{2n}$ and $y = x_{2n+1}$ in (e), we get

$$M(Px_{2n}, Qx_{2n+1}, kt) \geq \text{Min}\{M(Qx_{2n+1}, STx_{2n+1}, t), M(ABx_{2n}, STx_{2n+1}, t), M(Px_{2n}, ABx_{2n}, t)\}$$

$$= \text{Min}\{M(y_{2n+2}, y_{2n+1}, t), M(y_{2n}, y_{2n+1}, t), M(y_{2n+1}, y_{2n}, t)\}$$

$$\geq M(y_{2n}, y_{2n+1}, t). \; M(y_{2n+1}, y_{2n+2}, kt) \geq M(y_{2n}, y_{2n+1}, t).$$

Similarly, we have

$$M(y_{2n+2}, y_{2n+3}, kt) \geq M(y_{2n+1}, y_{2n+2}, t).$$

Thus, we have

$M(y_{n+1}, y_{n+2}, kt) \geq M(y_n, y_{n+1}, t)$ for n = 1, 2, ...

$$\begin{aligned} M(y_n, y_{n+1}, t) &\geq M(y_n, y_{n+1}, t/k) \\ &\geq M(y_{n-2}, y_{n-1}, t/k^2) \\ &\quad \dots \quad \dots \quad \dots \\ &\quad \dots \\ &\geq M(y_1, y_2, t/k^n) \to 1 \text{ as } n \to \infty, \end{aligned}$$

and hence $M(y_n, y_{n+1}, t) \to 1$ as $n \to \infty$ for any t > 0.

For each ε > 0 and t > 0, we can choose $n_0 \in N$ such that

$$M(y_n, y_{n+1}, t) > 1 - \varepsilon \quad \text{for all } n > n_0.$$

For m, n $\in$ N, we suppose m ≥ n. Then we have

$$\begin{aligned} M(y_n, y_m, t) &\geq M(y_n, y_{n+1}, t/m\text{-}n) * M(y_{n+1}, y_{n+2}, t/m\text{-}n) \\ &\quad * \dots * M(y_{m-1}, y_m, t/m\text{-}n) \\ &\geq (1 - \varepsilon) * (1 - \varepsilon) * \dots * (1 - \varepsilon)\ (m - n) \text{ times} \\ &\geq (1 - \varepsilon) \end{aligned}$$

and hence $\{y_n\}$ is a Cauchy sequence in X.

Since (X, M, *) is complete, $\{y_n\}$ converges to some point z $\in$ X. Also its subsequences converges to the same point i.e. z $\in$ X

i.e., $\{Qx_{2n+1}\} \to z$ and $\{STx_{2n+1}\} \to z$ (1)

$\{Px_{2n}\} \to z$ and $\{ABx_{2n}\} \to z$ (2)

Case I. Suppose AB is continuous.

Since AB is continuous, we have

$$(AB)^2 x_{2n} \to ABz \text{ and}$$
$$ABPx_{2n} \to ABz.$$

As (P,AB) is compatible, so by Proposition 2.2, $P(AB)x_{2n} \to ABz$.

Step 2. Put $x = ABx_{2n}$ and $y = x_{2n+1}$ in (e), we get

$$M(PABx_{2n}, Qx_{2n+1}, kt) \geq \text{Min}\{M(Qx_{2n+1}, ST x_{2n+1}, t), M(ABABx_{2n}, ST x_{2n+1}, t), M(PABx_{2n}, ABABx_{2n}, t)\}.$$

Taking $n \to \infty$, we get

M(ABz, z, kt) ≥ Min{M(z, z, t), M(ABz, z, t), M(ABz, ABz, t)}
M(ABz, z, kt) ≥ M(ABz, z, t)

Therefore, by using lemma 2.2, we get

ABz = z. (3)

Step 3. Put x = z and $y = x_{2n+1}$ in (e), we have

$$M(Pz, Qx_{2n+1}, kt) \geq \text{Min}\{M(Qx_{2n+1}, STx_{2n+1}, t), M(ABz, STx_{2n+1}, t), M(Pz, ABz, t)\}.$$

Taking $n \to \infty$ and using equation (1), we get

M(Pz, z, kt) ≥ Min {M(z, z, t), M(z, z, t), M(Pz, z, t)}.

i.e. M(Pz, z, kt) ≥ M(Pz, z, t).

Therefore, by using lemma 2.2, we get

Pz = z.

Therefore, ABz = Pz = z.

Step 4. Putting x = Bz and $y = x_{2n+1}$ in condition (e), we get

$$M(PBz, Qx_{2n+1}, kt) \geq \text{Min}\{M(Qx_{2n+1}, STx_{2n+1}, t), M(ABBz, STx_{2n+1}, t), M(PBz, ABBz, t)\}.$$

As BP = PB, AB = BA, so we have

P(Bz) = B(Pz) = Bz and (AB)(Bz) = (BA)(Bz) = B(ABz) = Bz.

Taking $n \to \infty$ and using (1), we get

M(Bz, z, kt) ≥ Min {M(z, z, t), M(Bz, z, t), M(Bz, Bz, t)}

i.e. M(Bz, z, kt) ≥ M(Bz, z, t).

Therefore, by using lemma 2.2, we get

Bz = z

and also we have

ABz = z

⇒ Az = z.

Therefore, $Az = Bz = Pz = z$. (4)

Step 5. As $P(X) \subset ST(X)$, there exists $u \in X$ such that

$$z = Pz = STu.$$

Putting $x = x_{2n}$ and $y = u$ in (e), we get

$$M(Px_{2n}, Qu, kt) \geq \text{Min}\{M(Qu, STu, t), M(ABx_{2n}, STu, t), M(Px_{2n}, ABx_{2n}, t)\}$$

Taking $n \to \infty$ and using (1) and (2), we get

$$M(z, Qu, kt) \geq \text{Min}\{M(Qu, z, t), M(z, z, t), M(z, z, t)\}$$

i.e. $M(z, Qu, kt) \geq M(z, Qu, t)$.

Therefore, by using lemma 2.2, we get

$$Qu = z.$$

Hence $STu = z = Qu$.

Since (Q, ST) is occasionally weakly compatible, so we have

$$QSTu = STQu.$$

Thus, $Qz = STz$.

Step 6. Putting $x = x_{2n}$ and $y = z$ in (e), we get

$$M(Px_{2n}, Qz, kt) \geq \text{Min}\{M(Qz, STz, t), M(ABx_{2n}, STz, t), M(Px_{2n}, ABx_{2n}, t)\}$$

Taking $n \to \infty$ and using (2) and step 5, we get

$$M(z, Qz, kt) \geq \text{Min}\{M(Qz, Qz, t), M(z, Qz, t), M(z, z, t)\}$$

i.e. $M(z, Qz, kt) \geq M(z, Qz, t)$.

Therefore, by using lemma 2.2, we get

$$Qz = z.$$

Step 7. Putting $x = x_{2n}$ and $y = Tz$ in (e), we get

$$M(Px_{2n}, QTz, kt) \geq \text{Min}\{M(QTz, STTz, t), M(ABx_{2n}, STTz, t), M(Px_{2n}, ABx_{2n}, t)\}.$$

As $QT = TQ$ and $ST = TS$, we have

$$QTz = TQz = Tz \text{ and } ST(Tz) = T(STz) = TQz = Tz.$$

Taking $n \to \infty$, we get

$$M(z, Tz, kt) \geq \text{Min}\{M(Tz, Tz, t), M(z, Tz, t), M(z, z, t)\}$$

i.e. $M(z, Tz, kt) \geq M(z, Tz, t)$.

Therefore, by using lemma 2.2, we get

$$Tz = z.$$

Now $STz = Tz = z$ implies $Sz = z$.

Hence $Sz = Tz = Qz = z$. (5)

Combining (4) and (5), we get

$$Az = Bz = Pz = Qz = Tz = Sz = z.$$

Hence, z is the common fixed point of A, B, S, T, P and Q.

Case II. Suppose P is continuous.

As P is continuous, $P^2x_{2n} \to Pz$ and $P(AB)x_{2n} \to Pz$.

As (P, AB) is compatible, so by proposition 2.2, $(AB)Px_{2n} \to Pz$

Step 8. Putting $x = Px_{2n}$ and $y = x_{2n+1}$ in condition (e), we have

$$M(PPx_{2n}, Qx_{2n+1}, kt) \geq \text{Min}\{M(Qx_{2n+1}, STx_{2n+1}, t), M(ABPx_{2n}, STx_{2n+1}, t), M(PPx_{2n}, ABPx_{2n}, t)\}.$$

Taking $n \to \infty$, we get

$$M(Pz, z, kt) \geq \text{Min}\{M(z, z, t), M(Pz, z, t), M(Pz, Pz, t)\}$$

i.e. $M(Pz, z, kt) \geq M(Pz, z, t)$.

Therefore by using lemma 2.2, we have

$$Pz = z.$$

Further, using steps 5, 6, 7, we get

$$z = Qz = STz = Sz = Tz.$$

Step 9. As $Q(X) \subset AB(X)$, there exists $u \in X$ such that

$$z = Qz = ABu.$$

Putting $x = u$ and $y = x_{2n+1}$ in (e), we get

$$M(Pu, Qx_{2n+1}, kt) \geq \text{Min}\{M(Qx_{2n+1}, STx_{2n+1}, t), M(ABu, STx_{2n+1}, t), M(Pu, ABu, t)\}$$

Taking $n \to \infty$, we get

$$M(Pu, z, kt) \geq \text{Min}\{M(z, z, t), M(z, z, t), M(Pu, z, t)\}$$

i.e. $M(Pu, z, kt) \geq M(Pu, z, t)$.

Therefore, by using lemma 2.2, we get

$$Pu = z.$$

Since $z = Qz = ABu$, so $Pu = ABu$.

Since (P, AB) is compatible, so by proposition 2.3, we have

$$PABu = ABPu$$

Or, $Pz = ABz$.

Also, $z = Bz$ follows from step 4. Thus, $z = Az = Bz = Pz$.

Therefore, $z = Az = Bz = Pz = Qz = Sz = Tz$, i.e. z is the common fixed point of the six maps A, B, S, T, P and Q in this case also.

Uniqueness : Let u be another common fixed point of A, B, S, T, P and Q.

Then $Au = Bu = Pu = Qu = Su = Tu = u$.

Put $x = z$ and $y = u$ in (e), we get

$$M(Pz, Qu, kt) \geq \text{Min}\{M(Qu, STu, t), M(ABz, STu, t), M(Pz, ABz, t)\}.$$

Taking $n \to \infty$, we get

$$M(z, u, kt) \geq \text{Min}\{M(u, u, t), M(z, u, t), M(z, z, t)\}.$$

i.e. $M(z, u, kt) \geq M(z, u, t)$.

Therefore by using lemma 2.2, we get

$$z = u.$$

Therefore, z is the unique common fixed point of self maps A, B, S, T, P and Q.

Remark 3.1. If we take $B = T = I$ in theorem 3.1, then the condition (b) is satisfied trivially and we get

Corollary 3.1. Let (X, M, *) be a complete Fuzzy metric space with continuous t-norm * and $t * t \geq t$, for all $t \in [0, 1]$ and let A, S, P and Q be mappings from X into itself such that the following conditions are satisfied:

(a) $P(X) \subset S(X)$, $Q(X) \subset A(X)$;

(b) either P or A is continuous;

(d) (P, A) is compatible and (Q, S) is occasionally weakly compatible;

(e) There exists $k \in (0, 1)$ such that $\forall x, y \in X$ and $t > 0$,

$$M(Px, Qy, kt) \geq \text{Min}\{M(Qy, Sy, t), M(Ax, Sy, t), M(Px, Ax, t)\}.$$

Then A, S, P and Q have a unique common fixed point in X.

Remark 3.2. In view of remark 3.1, corollary 3.1 is a generalization of the result of Singh et. al. [16] in the sense that condition of compatibility of the pairs of self maps has been restricted to compatible and occasionally weakly compatible self maps and only one of the mappings of the first pair is needed to be continuous.

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