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ANECESSARY AND REQUIRED CONDITION OF HAMBURGER MOMENT PROBLEM

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ABSTRACT

Objective: This present paper deals with necessary condition of Hamburger moment problem and polynomial which is not identically and non-negative sequence and semi-definite nature of a moment sequence.

Materials and Methods: If we suppose that $(s_n)_{n>=0}$ is a sequence of real numbers, the moment problem on I consists of solving the following three problems:

- There exists a positive measure on I with moment$(s_n)_{n>=0}$.
- This positive measure uniquely determined by the moments$(s_n)_{n>=0}$.
- The moment problem on [0,1) is referred to as Hausdroff moment problem and the moment problem on R is called Hamburger moment problem and the [0, ∞) is called Stieltjes moment problem.

Results: For n be an arbitrary non-negative integer, $\left|\int_{-T_1}^{T_2} t^n d\alpha(t) - \mu_n\right| < \in$ and sub-interval t^n in every sub-interval is not greater than $\in'$ such that $\left|\sum_{i=0}^{p-1} t_{i+1}^n [\alpha(t_{i+1}) - \alpha(t_i)] - \int_{-T_1}^{T_2} t^n d\alpha(t)\right| < \frac{\in}{2}$. The function V(t) in terms of operator M is ifα(t) had infinitely many points of non-decrease, then for every positive polynomial P(t) not identically zero,

$$|M[V(t)] - \mu_n| \leqq \in' (\mu_0 + \mu_{2m}) = \frac{\in}{2}. \quad M[P(t)] = \sum_{k=0}^{n} \alpha_k \mu_k = \int_{-\infty}^{\infty} P(t) d\alpha(t) \geqq 0.$$

Conclusion: For increasing function α(t) has a finite number of points of non-increase. Every non-negative sequence is either definite or semi-definite.

Keywords: Hamburger moment problem, Moment sequence, monotonic sequence,Rational number,Stieltjes integral.

INTRODUCTION

The Moment polynomials

To obtain conditions on the sequence $\{\mu_n\}_0^\infty$ which will initialize that the corresponding Hamburger moment problem will have at least one increasing solution. FromM. Riesz method of moment of a polynomial [1],we define moment of a polynomial and the corresponding sequence.
Definition: The moment M[P(t)] of a polynomial

$P(t) = \sum_{k=0}^{n} \alpha_k t^k$ and the correspondingsequence $\{\mu_n\}_0^\infty$ is $M[P(t)] = \sum_{k=0}^{n} \alpha_k \mu_k$.

The Stieltjes moment problem

The Stieltjes moment problem can also be applied as a special case of the Hamburger problem [2-6], in which a significant and required condition is that there should existincreasing function $\alpha(t)$ such that

$$\mu_n = \int_0^\infty t^n d\alpha(t) \qquad (n = 0, 1, 2, ...), \quad (1)$$

The integrals are all converging;thus,the sequences $\{\mu_n\}_0^\infty$ and $\{\mu_n\}_1^\infty$ should be non-negative, or in the quadratic formsare as:

$$\sum_{i=0}^{n}\sum_{j=0}^{n} \mu_{i+j}\xi_i\xi_j \qquad (n = 0, 1, 2, ...) \qquad (2)$$

$$\sum_{i=0}^{n}\sum_{j=0}^{n} \mu_{i+j+1}\xi_i\xi_j \qquad (n = 0, 1, 2, ...) \qquad (3)$$

should be non-negative (definite or semi-definite).

The Hamburgerproblem

The Hamburger and Stieltjes problems for the case in which $\alpha(t)$ is of closely related variation on the appropriate infinite interval. R.P. Boas [7-11] has observed that in this case there is hardly any problem, as every sequence leads to a soluble Stieltjes or Hamburger problem if we consider any function of closely related variation as a solution.

We provide the proof to this problem by initializing the Stieltjes case.

The equations

$$\mu_n = \int_0^\infty t^n d\alpha(t)$$
(n = 0, 1, 2, ...)

has a solution $\alpha(t)$ of closely related variation in which, $\int_0^\infty |d\alpha(t)| < \infty$.

We form the other two sequences $\{\lambda_n\}_0^\infty, \{\nu_n\}_0^\infty$, as:

$$\mu_n = \lambda_n - \nu_n \quad (4)$$

$$\lambda_n = \int_0^\infty t^n d\beta(t) \quad (5)$$

$$\nu_n = \int_0^\infty t^n d\gamma(t) \quad (n = 0, 1, 2, ...). \quad (6)$$

MATERIALS AND METHODS

Let I $\subseteq$ R be an interval. For a positive measure μ on Ithe nth moment is defined as $\int_I x^n d\mu(x)$ provided the integral exists. If we suppose that $(s_n)_{n>=0}$ is a sequence of real numbers, the moment problem on I consists of solving the following three problems:

(I) There exists a positive measure on I with moment$(s_n)_{n>=0}$.

(II) This positive measure uniquely determined by the moments$(s_n)_{n>=0}$.

When μ is a positive measure with moments $(s_n)_{n>=0}$ we say that μ is a solution to the moment problem. If the solution to the moment problem is unique, the moment problem is called determinate, otherwise the moment problem is said to be indeterminate.

(III) All positive measure on I with moments $(s_n)_{n>=,0}$can be uniquely describedby giving historical reason i.e. The moment problem on [0,1) is referred to as the Hausdroff' moment problem and the moment problem on R is called the Hamburger moment problem and thus [0, ∞) is called the Stieltjes moment problem.

The Hamburger moment problem

Axiom 2.1(a): A necessary and required condition that there should be an existence of at least one increasing function $\alpha(t)$ in a way as

$$\mu_n = \int_{-\infty}^\infty t^n d\alpha(t) \quad (n = 0, 1, 2, ...) \quad (7)$$

all the integrals converging, is that the sequence $\{\mu_n\}_0^\infty$ should be non-negative.

Let $\alpha(t)$ be an increasing solution of equations (7) where P(t) is an arbitrary positive polynomial,

$$P(t) = \sum_{k=0}^n \alpha_k t^k \cdot$$

As $M[P(t)] = \sum_{k=0}^n \alpha_k \mu_k = \int_{-\infty}^\infty P(t) d\alpha(t) \geqq 0'$

the sequence $\{\mu_n\}$ is non-negative. Hence, the condition is significantly necessary [12-15]. Correspondingly, let the sequence $\{\mu_n\}_0^\infty$ be non-negative.

We will initialize an increasing solution $\alpha(t)$ of (7). It is explained and at the rational points by the equations-

$$\alpha(\xi_m) = M[h_m(t)]$$
(m = 0, 1, 2, ...).

Then if $\xi_i < \xi_j$, we concluded from the non-negative nature of M

$$h_i(t) \leqq h_j(t)$$

$$\alpha(\xi_i) \leqq \alpha(\xi_j).$$

Now, $\alpha(t)$ is an increasing so, as far as it has been explained and defined.

Let η be an arbitrary irrational number in order to complete the definition, and even suppose,

$$\overline{\zeta} = \underset{\xi_m > \eta}{\text{l.b.}} \alpha(\xi_m)$$

$$\underline{\zeta} = \underset{\xi_m < \eta}{\text{u.b.}} \alpha(\xi_m)$$

$$\alpha(\eta) = \frac{\overline{\zeta} + \underline{\zeta}}{2}.$$

As, $\alpha(t)$ is an increasing on the rational points it is clearly stated that $\underline{\zeta}$ is lesser than $\overline{\zeta}$ and that $\alpha(t)$ as now completely defined is increasing.

Suppose, n be an arbitrary non-negative integer. We aim to prove that for this n,

$$\mu_n = \int_{-\infty}^\infty t^n d\alpha(t).$$

Though, $\alpha(t)$ is an increasing, it will be enough to provide that to an arbitrary non-negative $\in$ there converges a positive T_0 in such a way that for every pair of rational numbers T_0 is lesser than T_1 and T_2

$$\left| \int_{-T_1}^{T_2} t^n d\alpha(t) - \mu_n \right| < \in \quad (8)$$

Suppose, m be an integer such that $2m > n$. Set

$$\in' = \frac{\in}{2(\mu_0 + \mu_{2m})}$$

the sub-interval so small that the t^n in every sub-interval is not greater than $\in'$,such as;

$$\left| \sum_{i=0}^{p-1} t_{i+1}^n [\alpha(t_{i+1}) - \alpha(t_i)] - \int_{-T_1}^{T_2} t^n d\alpha(t) \right| < \frac{\in}{2} \quad (9)$$

It is quite possible and identified by the uniform continuity of t^n and by definition of Stieltjesintegral.

Let us discuss the function V(t) as follows:

$$V(t)=0$$
$$(t \leqq -T_1, t>T_2)$$
$$=t_{i+1}^n$$
$$(t_i<t\leqq t_{i+1}; i=0,1,\ldots,p-1).$$

Thus

$$V(t)=\sum_{i=0}^{p-1} t_{i+1}^n\left[h_{k_{i+1}}(t)-h_{k_i}(t)\right]$$

And,

$$\left|V(t)-t^n\right| \leqq \in' t^{2m}$$
$$(t\leqq -T_1, t>T_2)$$

Using (7). we havealso;

$$\left|V(t)-t^n\right| \leqq \sum_{i=0}^{p-1} \in' \left[h_{k_{i+1}}(t)-h_{k_i}(t)\right]=\in'$$
$$(-T_1<t\leqq T_2),$$

so that;

$$\left|V(t)-t^n\right| \leqq \in' + \in' t^{2m}$$
$$(-\infty<t<\infty).$$

Here, V(t) is associated with one of the sets E_i, because the operator M is non-negative and distributive as applicable to V(t) which is given as;

$$\left|M[V(t)]-\mu_n\right| \leqq \in' (\mu_0+\mu_{2m})=\frac{\in}{2} \quad (10)$$

But,

$$M[V(t)]=\sum_{i=0}^{p-1} t_{i+1}^n\left[\alpha(t_{i+1})-\alpha(t_i)\right].$$

By the combination of equations (9) and (10) we get (8), which is the required goal to be achieved.

RESULTS

Polynomial which is not identically zero

The non-negative sequence is classified as: the non-negative definite and non-negative semi-definite sequences [16-19].
Observation: A non-negative sequence is either definite or semi-definite.

We would like to characterize these two cases by situations on thesequence $\{\mu_n\}_0^\infty$. Werequire the following Lemma and axioms.

Lemma 3.1 (a): Every real positive polynomial is the sum of the squares of two real polynomials.

Since, the polynomial is actually real so its imaginary roots, if any, occur in conjugate imaginary pairs then; the degree of the polynomial must be even or zero. Thus, its factored form must be

$$\prod_{i=0}^{n}\left[(x-\alpha_i)^2+\beta_i^2\right]\prod_{i=0}^{m}(x-\gamma_i)^2,$$

But here $\alpha_i, \beta_i, \gamma_i$ are real numbers. Following the trivial identity,

$$|x_1+iy_1|^2|x_2+iy_2|^2=|(x_1+iy_1)(x_2+iy_2)|^2,$$

We get,

$$\left[(x-\alpha_i)^2+\beta_i^2\right]\left[(x-\alpha_j)^2+\beta_j^2\right]=\left[(x-\alpha_i)(x-\alpha_j)-\beta_i\beta_j\right]^2$$
$$+\left[\beta_j(x-\alpha_i)+\beta_i(x-\alpha_j)\right]^2.$$

This is the product of the total of two squares by the total of two squares is the total of two squares.The repeated application of this outcome proves the lemma.

Axiom 3.1(b): A significantly required and necessary condition that the sequence $\{\mu_n\}_0^\infty$ should be semi-definite is that the quadratic forms

$$\sum_{i=0}^{n}\sum_{j=0}^{n}\mu_{i+j}\xi_i\xi_j \qquad (n = 0, 1, 2, \ldots) \qquad (11)$$

could be semi-definite.

Let the quadratic forms of equation (11) be all non-negative (definite of semi-definite). Suppose P(t) be an arbitrary positive polynomial. Using lemma, it is the total of the squares of two other polynomials $P_1(t)$ and $P_2(t)$:

$$P_1(t)=\sum_{i=0}^{n}\alpha, t^i,$$
$$P_2(t)=\sum_{i=0}^{m}\beta_i t^i,$$

so

$$M[P(t)]=\sum_{i=0}^{n}\sum_{j=0}^{n}\alpha_i\alpha_j\mu_{i+j}+\sum_{i=0}^{m}\sum_{j=0}^{m}\beta_i\beta_j\mu_{i+j},$$

is taken as positive by hypothesis. Hence, $\{\mu_n\}_0^\infty$ is non-negative. Now, Let the forms of equation (11) are non-negative definite i.e. no form can become zero unless all of its variables are disappeared. Then, if the polynomial P(t) mentioned above is not identically zero, the α_i and β_i are noted as all zero from where we observe that M[P(t)] is actually greater than zero, and the sequence $\{\mu_n\}$ is non-negative definite.

Correspondingly, let $\{\mu_n\}_0^\infty$ be a non-negative definite sequence. Let n be an arbitrary non-negative integer and $\xi_0, \xi_1, \ldots, \xi_n$ arbitrary constants are not all zero.

To Prove

$$\sum_{i=0}^{n}\sum_{j=0}^{n}\mu_{i+j}\xi_i\xi_j > 0.$$

This has been followed from the definition of a non-negative definite sequence, since the polynomial $\left[\sum_{i=0}^{n}\xi_i t^i\right]^2$ is positive and not identically zero. If the sequence $\{\mu_n\}_0^\infty$ is only to be non-negative then we get;

$$M\left[\left(\xi_0 + \xi_1 t + \ldots + \xi_n t^n\right)^2\right] = \sum_{i=0}^{n}\sum_{j=0}^{n}\mu_{i+j}\xi_i\xi_j \geqq 0,$$

which is because of the formsof equation (11) are at least non-negative (definite or semi-definite). As a non-negative sequence or form is either definite or semi-definite, thus it leads to the completion of theorem.

Corollary: Monotonic sequence is a non-negative sequence only if it is fully complete.

The completely monotonic sequence $\{\mu_n\}_0^\infty$ has the integral representation

$$\mu_n = \int_0^1 t^n d\alpha(t) \qquad (n = 0, 1, 2, \ldots) \qquad (12)$$

Where $\alpha(t)$ is increasing. Hence the quadratic forms are:

$$\sum_{i=0}^{n}\sum_{j=0}^{n}\mu_{i+j}\xi_i\xi_j = \int_0^1\left(\sum_{i=0}^{n}t^i\xi_i\right)^2 d\alpha(t)$$

(n = 0, 1, 2, ...)

which are actually non-negative.

Findingdomain

We next examine the effect of the definite or semi-definite nature of a moment sequence on its integral representation.

Axiom 3.2 (a) A required and significant condition that there could be an existence of increasing function $\alpha(t)$ with infinitely many points of non-decrease (with a finite number of points of increase) such as:

$$\mu_n = \int_{-\infty}^{\infty} t^n d\alpha(t) \qquad (n = 0, 1, 2, \ldots) \qquad (13)$$

Given above is that the sequence $\{\mu_n\}_0^\infty$ could be non-negative definite (semi-definite).

To show : If there is an existence of an increasing solution $\alpha(t)$ of (13) with but a finite number of points of non-decrease $t_0, t_1, \ldots, t_m$, then the sequence $\{\mu_n\}_0^\infty$ is non-negative semi-definite. This has been followed as;

$$M\left[(t-t_0)^2(t-t_1)^2\ldots(t-t_m)^2\right]$$

$$= \int_{-\infty}^{\infty}(t-t_0)^2(t-t_1)^2\ldots(t-t_m)^2 d\alpha(t) = 0.$$

Correspondingly, it has a representation (13) with an increasing $\alpha(t)$ by axiom 3.1(a) only if the sequence $\{\mu_n\}_0^\infty$ is non-negative semi-definite. If $\alpha(t)$ had infinitely many points of non-decrease, then for every positive polynomial P(t) not identically zero.

$$M[P(t)] = \int_{-\infty}^{\infty} P(t) d\alpha(t) > 0,$$

on contrary tohypothesis. Thus $\alpha(t)$ has finite number of points of non-increase.

Every non-negative sequence is either definite or semi-definite. Hence the proof.

Axiom 3.2(b): It gives us another proof of corollary. According to axiom for a completely monotonic sequence has the representation (13) with $\alpha(t)$ an increasing in (0, 1) and constant everywhere.

Some other result of the same nature but less useful, as it has no contradictory for the non-negative semi-definite case is taken.

Axiom 3.2(c): A significant and required condition that equations (13) should have an increasing solution $\alpha(t)$ with infinitely many points of increase such as;

$$\mu_0 > 0, \quad \begin{vmatrix}\mu_0 & \mu_1\\ \mu_1 & \mu_2\end{vmatrix} > 0, \quad \begin{vmatrix}\mu_0 & \mu_1 & \mu_2\\ \mu_1 & \mu_2 & \mu_2\\ \mu_2 & \mu_3 & \mu_4\end{vmatrix} > 0\ldots, \qquad (14)$$

Using thisaxiom, it has been followed that significant and required condition from algebra of a quadratic form should be non-negative definite.

It is clearly observed thatif equations (13) has an increasing solution $\alpha(t)$ with a finite number of points of non-decrease then the determinants (14) are non-negative or zero. But conversely, it contradicts the statement that equations (13) have a positive solution whenever determinants (14) are positive.

For instance: if

$$\mu_n = \begin{cases}1 & (n = 0,1,2,3)\\ 0 & (n = 4,5,\ldots),\end{cases}$$

We get,

$$\mu_0 > 0, \quad \begin{vmatrix}\mu_0 & \mu_1\\ \mu_1 & \mu_2\end{vmatrix} = 0$$

$$\begin{vmatrix}\mu_0 & \mu_1 & \mu_2\\ \mu_1 & \mu_2 & \mu_3\\ \mu_2 & \mu_3 & \mu_4\end{vmatrix} = 0 \qquad \begin{vmatrix}\mu_0 & \mu_1 & \mu_2 & \mu_3\\ \mu_1 & \mu_2 & \mu_3 & \mu_4\\ \mu_2 & \mu_3 & \mu_4 & \mu_5\\ \mu_3 & \mu_4 & \mu_5 & \mu_6\end{vmatrix} > 0,$$

as given above with all progressive determinants zero. Though, the quadratic form

$$\sum_{i=0}^{2}\sum_{j=0}^{2}\xi_i\xi_j\mu_{1+1}$$

is neither non-negative definite nor non-negative semi-definite. For it reduces to –1 when,

$$\xi_0 = -1, \quad \xi_1 = 0, \quad \xi_2 = 1,$$

CONCLUSION

This research paper focuses on necessary and sufficient condition for polynomial which is not identically zero. It deals with Stieltjes moment problem andHamburger problem. In discussing the moment problemsthere requires the finding domain and the condition for solving moment, determining and non-determining solution.

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