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LIMIT INTERCHANGE IN INTEGRATION AND SUMMATION BY USING L'HOSPITAL'S RULE

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Abstract : It is known that L'Hospital's Rule is one of the important results used for obtaining the limits of those functions which are of the type $\frac{f(x)}{g(x)}$ at a given point 'a' where $\frac{f(a)}{g(a)}$ may take an indeterminate form like $\frac{0}{0}, \frac{\infty}{\infty}$, or $\frac{k}{\infty}$. The main purpose of this paper is to illustrate some problems and results in advanced calculus with the help of this rule and examples .we shall show that there are some sequences of functions for which the integral of the limit is equal to limit of the integral .We also illustrate through examples the L'Hospital's Rule fails in the limit interchange process if a sequence under consideration is not uniformly convergent.

INTRODUCTION

L'Hospital's Rule [3] is one of the important results for obtaining the limits of those rational functions of the type $\frac{f(x)}{g(x)}$ at a given point 'a', where $\frac{f(a)}{g(a)}$ may take an indeterminate form like $\frac{0}{0}, \frac{\infty}{\infty}$, or $\frac{k}{\infty}$. Even if a student does not know the details of the proof of this result he/she can easily obtain the limits of many functions of the type $\frac{f(x)}{g(x)}$ at a given number 'a' where $\frac{f(a)}{g(a)}$ may have any one of the above mentioned indeterminate forms.

The study of interchange of limit operation is one of the major concepts in Mathematical Analysis. Taking the limit inside the integral is not always allowed. There are several theorems that allow us to do so. The major ones being Lebesgue dominated convergence theorems that allow us to do so. The major ones being Lebesgue dominated convergence and monotone convergence theorems. The uniform convergence theorem is a special case of dominated convergence theorem. By means of examples we are going to show that limit processes cannot in general be interchanged without affecting the result. In fact the limit interchange process can also be explained through the use of L'Hospital's Rule. In this paper the following problems and results in advanced calculus are illustrated with the help of examples and this rule.

(i) Finding the second derivative of a given function if it exists,

(ii) If $\{X_{m,n}\}$ is a double sequence of real numbers depending on the positive integers

m, n then can we say that

$$\lim_{n\to\infty}\lim_{m\to\infty} x_{m,n} = \lim_{m\to\infty}\lim_{n\to\infty} x_{m,n}$$

(iii) The integral of the limit of a sequence $\{f_n\}$ of real valued integrable functions

coincides with the limit of the integral of $\{f_n\}$ under certain conditions, and

(iv) The integral of the sum of infinite number of real valued integrable functions

coincides with the sum of their integrals under certain conditions.

Further counter examples are also given to show that the application of

L'Hospital Rule in the limit interchange process fails, if the sequence of functions under consideration is not uniformly convergent.

PRELIMINARIES

For the sake of completeness we given below some necessary definitions and known results from Real Analysis.

Definition 2.1: We say that a sequence $\{f_n(x)\}$ real valued functions defined on the closed interval [a, b] converges uniformly to a function $f(x)$ if for every $\in > 0$ there is a positive integer N such that n ≥ N implies $|fn(x)\} - f(x)|$ $\leq \in$ for all x in [a, b]

Definition 2.2: We say that a series $\sum_{n=1}^{\infty} fn(x)$ of real valued functions defined on I = [a, b] converges uniformly on I if the sequence $s_n(x)$of partial sums defined by

$\sum_{i=1}^{\infty} f_i(x) = s_n(x)$ Converges uniformly on I.

Remark 2.1: It is clear that every uniformly convergent sequence is point wise convergent, but the converse not always true. For example the sequence $\left\{\frac{1}{\left(1+\frac{x}{n}\right)^{n}}\right\}$ is uniformly convergent on [0,1] and hence it is convergent at every point of [0,1]. Similarly it can also be shown that the sequence $\{f_{n}(x)\}$ defined on [0, 1] by the relation

$\{f_{n}(x)\}$ =2n, if 1/n ≤ x ≤ 2/n,

= 0, otherwise , is not uniformly convergent.

Definition 2.3: A double sequence $x_{m,n}$ of real numbers, m, n = 1, 2, 3… is said to be convergent if

$$\lim_{n\to\infty}\lim_{m\to\infty} x_{m,n} = \lim_{m\to\infty}\lim_{n\to\infty} x_{m,n}$$

(2.1)

Note that on the left side of (2.1), we first let m→∞, then n→∞; on the right side of (2.1) n→∞ first and, then m→∞.

Definition 2.4: Let E C [a, b]. An outer measure of E denoted by m̄ (E) is defined as m̄ (E) = g.l.b. |G| where g.l.b. is taken over all open sets G which contain E, where |G| = $\sum_{n}|I_{n}|$, ($|I_{n}|$denotes the length of the interval In). Similarly the inner measure m_(E) of E is defined as m_(E) = l.u.b.|F| where l.u.b. is taken over all closed setsf contained in E.

Definition 2.5: A set E is said to be measurable if m̄ (E) = m_(E). In this case we define the measure of E as m(E) = m̄ (E) = m_(E).

Definition 2.6: A statement S is said to hold almost everywhere on a set E C [a,b] if the set of those points of E at which the statement does not hold is of measure zero. In this case express this idea by writing 'S holds a.e. on E'.

We now state L'Hospital Rule in the form of a theorem.

Theorem 2.1 (L' Hospital's Rule):

Let f and g be two real valued differentiable functions in the open interval (a, b) and g′(x) ≠ 0 for all x ϵ (a, b), & $-\infty \leq$ a $<$ b $\leq \infty$.

Let

$$\lim_{x\to a}\frac{f'(x)}{g'(x)} = L \; If \; f(x) \to 0 \; and \; g(x) \to 0 \; as \; x \to a \; or \; if \; g(x) \to \infty \; as \; x \to a$$

, then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)} = L.$

The following theorems are related to uniform convergence and limit interchange in the process of integration.

Theorem 2.2:

Let $\{f_{n}(x)\}$ be a sequence of Riemann integrable functions on [a, b] and let f_{n}→f

uniformly on [a, b]. Then f is Riemann integrable on [a, b], and

$$\int_{a}^{b} f(x)\,dx = \lim_{n\to\infty}\int_{a}^{b} f_{n}(x)\,dx\,,$$

$$\int_{a}^{b} lim_{n\to\infty} f_{n}(x)dx = lim_{n\to\infty}\int_{a}^{b} f_{n}(x)dx \qquad (2.2)$$

Theorem 2.3:

Let $\{f_n(x)\}$be a sequence of Riemann integrable functions on [a, b] and let$f(x) = \sum_{n=1}^{\infty} f_{n}(x) \; (a \leq x \leq b)$ be a convergent series converging uniformly on [a, b], then

$$\int_{a}^{b} f(x)\,dx = \int_{a}^{b}\sum_{n=1}^{\infty} f_{n}(x)dx = \sum_{n=1}^{\infty}\int_{a}^{b} f_{n}(x)dx$$

(2.3)

In other words, the series may be integrated term by term.

Theorem 2.4(Lebesgue's Monotone Convergence Theorem):

Let$\{f_{n}\} : n$ = 1, 2, 3 , … be a sequence of measurable functions such that

$$0 \leq f_{1}(x) \leq f_{2}(x) \leq \cdots \qquad (x \in E)$$

(2.4)

Where E is a given measurable set.

Let $\{f_{n}\} \to f(x) \;\; (x \in E), as \; n \to \infty$

Then

$$\int_{a}^{b}\lim_{n\to\infty} f_{n}(x)dx = \lim_{n\to\infty}\int_{a}^{b} f_{n}(x)dx$$

(2.5)

Theorem 2.5 (Lebesgue's Dominated Convergence Theorem):

Let $\{f_n\} : n$=1,2,3,... be a sequence of Lebesgue integrable functions on the interval I = [a, b], such that lim n→∞ $f_n(x)$ = f(x) almost everywhere on I. Suppose there exists a Lebesgue integrable function g on I such that $|f_n(x)| \leq g(x)$ almost every where on I. Then f is also Lebesgue integrable function on I and

$$\lim_{n\to\infty}\int_a^b f_n(x)dx = \int_a^b \lim_{n\to\infty} f_n(x)dx$$

3. SECOND DERIVATIVE BY L'HOSPITAL'S RULE

Theorem 3.1: If the second derivatives of a function f(x) defined on I = (a, b) exists at a point $x \in I$ then

$$f^n(x) = \lim_{h\to 0}\frac{f(x+h)+f(x-h)-2f(x)}{h^2}\ ,\ x \in I \quad (3.1)$$

Proof:

Applying L'Hospital's Rule two times we observe that

$$\lim_{h\to 0}\frac{f(x+h)+f(x-h)-2f(x)}{h^2}$$

$$= \lim_{h\to 0}\frac{f'(x+h)-f'(x-h)}{2h}$$

$$= \lim_{h\to 0}\frac{f''(x+h)-f''(x-h)}{2} = f''(x)$$

The proof is complete. Let us illustrate this theorem in finding the second derivative of x^n and log x, where n is a positive integer.

Let $f(x) = x^n$ then using the above theorem and L'Hospital's Rule we get,

$$f''(x) = \lim_{h\to 0}\frac{(x+h)^n+(x-h)^n-2x^n}{h^2}$$

$$= \lim_{h\to 0}\frac{2[n_{C_1}x^{n-1}h+n\,c_2x^{n-2}h^2+\dots+\dots]}{h^2}$$

$$= \lim_{h\to 0}\frac{2[n_{C_1}x^{n-1}+n\,c_2x^{n-2}\,2h+\dots+\dots]}{2h}$$

$$= \lim_{h\to 0}\frac{2[n_{C_2}x^{n-2}\times 2+n\,c_3x^{n-3}\times 6h+\dots+\dots]}{2}$$

$$= 2\times n\,C_2x^{n-2} = n(n-1)x^{n-2}$$

Similarly one can show that the second derivative of log x is $\frac{-1}{x^2}$ by using the above theorem.

INTERCHANGE OF LIMITS IN A DOUBLE SEQUENCE

Under certain conditions it is known that if $\{x_{m,n}\}$ is a double sequence of real numbers depending on the positive integer's m, n then

$$\lim_{n\to\infty}\lim_{m\to\infty}x_{m,n} = \lim_{m\to\infty}\lim_{n\to\infty}x_{m,n}$$

Let us illustrate this result for the sequence $x_{m,n} = \frac{3m}{m^2+5n}$

$$\lim_{n\to\infty}\lim_{m\to\infty}x_{m,n} = \lim_{m\to\infty}\lim_{n\to\infty}\frac{3m}{m^2+5n}$$

$$= \lim_{m\to\infty}\lim_{n\to\infty}\frac{3}{2m} = \lim_{n\to\infty}0 = 0 \text{ using L'Hospital's Rule} \quad (4.1)$$

On the other hand we observe that

$$\lim_{m\to\infty}\lim_{n\to\infty}x_{m,n} = \lim_{m\to\infty}\lim_{n\to\infty}\frac{3m}{m^2+5n}$$

$= \lim_{m\to\infty}\frac{0}{5} = 0$ (using L'Hospital's Rule).

This shows that the two sided limits are equal.Consider now the sequence $x_n = \frac{3m+2}{3m+5n}$ and see what the two sided limits are for this.

$$\lim_{n\to\infty}\lim_{m\to\infty} x_{m,n} = \lim_{n\to\infty}\lim_{m\to\infty}\frac{3m+2}{2m+5n}$$

$$= \lim_{n\to\infty}\frac{3}{2} = \frac{3}{2}$$ (Using L'Hospital's Rule) (4.2)

On the other hand we observe that

$$\lim_{m\to\infty} x_{m,n} = \lim_{n\to\infty}\lim_{m\to\infty}\frac{3m+2}{2m+5n}$$

$$= \lim_{m\to\infty}\frac{0}{5} = 0$$ (using L'Hospital's Rule) (4.3)

This shows that the two limits are not always equal even if we apply L'Hospital's Rule..

5. LIMIT INTERCHANGE IN IN TEGRATION

In this section we shall show that there are some sequences of functions for which the integral of the limit is equal to the limit of the integral, i.e. the relation (2.2) holds. We also illustrate through examples that L'Hospital's Rule fails in the limit interchange process if a sequence under consideration is not uniformly convergent. We begin with one example in which L'Hospital's Rule is used.

Example 5.1:

Consider the sequence $f_n(x)$ defined by the relation

$$f_n(x) = \frac{2nx+3}{n+1}, 0 \le x \le 1, n = 1,2,3$$

Applying L'Hospital's Rule we observe that of examples

$$\lim_{n\to\infty} f_n(x)dx = \lim_{n\to\infty}\frac{2nx+3}{n+1}$$

$$= \lim_{n\to\infty}\frac{2x+0}{1+0} = 2x$$

(5.1)

Further we also observe that

$$\mathrm{Lim}_{n\to\infty}|f_n(x) - 2x| = \lim_{n\to\infty}\left|\frac{2nx+3}{n+1} - 2x\right| \,| = \lim_{n\to\infty}\left|\frac{3-2x}{n+1} - 2x\right|$$

= 0 (using L'Hospital's Rule) (5.2)

Hence the sequence $\{f_n(x)\}$ converges to the function 2x on [0, 1]. Now again by using L'Hospital's Rule we show that the relation (2.2) holds for this sequence.

$$\lim_{n\to\infty}\int_0^1 f_n(x)dx = \lim_{n\to\infty}\int_0^1 \frac{2nx+3}{n+1}\,dx$$

$$= \lim_{n\to\infty}\left[\frac{(2nx+3)^2}{4n(n+1)}\right]$$

$$= \lim_{n\to\infty}\frac{4n^2+12n}{4n^2+4n}$$

$$= \lim_{n\to\infty}\frac{n+3}{n+1} = 1 \qquad (5.3)$$

On the other hand

$$\int_0^1 \lim_{n\to\infty} f_n(x)dx = \int_0^1 \lim_{n\to\infty} 2x\,dx = 1$$

(5.4)

Thus the relation (2.2) holds for the sequence we have considered.

Example 5.2:

Consider the sequence $\{f_n(x)\}$ defined by the relation $f_n(x) = nx\,(1-x^2)^n$,

$0 \le x \le 1$. It is easy to verify that this sequence is point wise convergent on the given interval. For this sequence we observe that

$$0 = \int_0^1 0\,dx = \int_0^1 0 \lim_{n\to\infty} nx(1-x^2)^n\,dx$$

(5.5)

$$= \lim_{n\to\infty}\int_0^1 0 \lim_{n\to\infty} nx = \lim_{n\to\infty}\frac{n}{2n+2} = \frac{1}{2}$$

Thus we have an absurd result 0 = ½.

Here we have assumed that integral of a limit is equal to the integral for the given sequence of functions. In fact this result is true only when the sequence of functions under consideration is uniformly convergent. Hence L'Hospital's Rule can be used for every uniformly convergent sequence of functions as far as the limit interchange process is concerned.

6. LIMIT INTERCHANGE AND SUMMATION

In this section we illustrate the result (2.3) given in section 2, by considering two examples. The first example is taken from [1].

Example 6.1:

Let n be a positive integer and a be any real number. They by an applicationof L'Hospital's Rule we observe that

a n = n×a = a+a+⋯+a (n terms)

$$= \lim_{r\to 1}[ar + ar^2 + ar^3 + \cdots + ar^{n-1}]$$

$$= \lim_{r\to 1}\frac{a[r^n-1]}{r-1}$$

$$= \lim_{r\to 1}[a \times nr^{n-1}]/1$$

$= a.n$ (using L'Hospital's Rule).

This simple and trivial example shows that the limit of a sum is equal to the sum of the limits.

Example 6.2:

Consider the series defined by the relation

$$\sum_{n=0}^{\infty}(-1)^n t^n = 1 - t + t^2 - t^3 + \cdots$$

For this series let us show that the relation (2.3) mentioned in Section 2 holds

On the interval [0,x]. In others words we shall show that

$$\int_0^x \sum_{n=0}^{\infty}(-1)^n t^n \, dt = \sum_{n=0}^{\infty}\int_0^x (-1)^n t^n \, dt$$

(6.1)

For this purpose we shall use the following known result.

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \log(1+x)$$

(6.2)

$$\int_0^x \sum_{n=0}^{\infty}(-1)^n t^n \, dt = \int_0^x (1 - t + t^2 - t^3 + \cdots)dt$$

$$= \int_0^x \lim_{n\to\infty}(1 - t + t^2 - t^3 + \cdots + (-t)^{n-1}dt$$

$$= \int_0^x \lim_{n\to\infty}\frac{1-(-t)^n}{1+t}dt$$

$$= \int_0^x \frac{1}{1+t} \quad (since\ |t| \leq 1$$

= log (1+x). (6.3)

On the other hand, we see that the right hand side of (6.1) is

$$\sum_{n=0}^{\infty}\int_0^x (-1)^n t^n \, dt = \sum_{n=0}^{\infty}[\frac{t^{n+1}(-1)^n}{n+1}]_0^x$$

$$= [x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

= log(1+x). (6.4)

From (6.3) and (6.4) we conclude that the relation (6.1) holds true. This is so because the series of functions that we have considered is uniformly convergent. Such a result does not hold for the series which is not uniformly convergent.

For example it can be shown that the series $\sum_{n=1}^{\infty}(-1)^n\frac{x^2+n}{n^2}$,for $x \in [a, b]$

is not uniformly convergent if the interval [a ,b] is unbounded.

CONCLUSION

We have seen that some results in Advanced Calculus can be explained with the help of L'Hospital's Rule and examples to the undergraduate students even if they don't know the concept of uniform convergence and advanced theorems in Real Analysis.

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